MODIFIED ELLIPTIC GAMMA FUNCTIONS AND $6d$ SUPERCONFORMAL INDICES

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ABSTRACT. We construct a modified elliptic gamma function of order two which is well defined when one of the base parameters lies on the unit circle, e.g. $|q| = 1$. Possible applications of this function in the context of $6d$ superconformal indices is discussed. A $6d/4d$ boundary theory with $SP(2N)$ gauge group is proposed whose superconformal index is $W(E_7)$-group invariant. For $N = 1$ this symmetry reduces to the symmetry transformation for an elliptic analogue of the Euler-Gauss hypergeometric function. A $5d/3d$ boundary field theory partition function with the same symmetry is discussed as well.

1. INTRODUCTION

Six dimensional superconformal field theories currently form an active research field (see, e.g., [1] and references therein). As claimed by Moore [1], these theories should form a gold mine for experts in special functions as a source of amazing identities, which is just one of many important potential mathematical outputs from them. At first unexperienced glance this statement may sound weird, but the author agrees with it. Indeed, a principally new class of special functions called elliptic hypergeometric integrals has been discovered in [2]. It came as a big surprise to mathematicians since it was tacitly assumed that q-hypergeometric functions form the top level special functions of hypergeometric type with nice exact formulae [3]. Some particular examples of such integrals were interpreted as wave functions or normalizations of wave functions in specific elliptic multiparticle quantum mechanical systems [2]. Recently it was shown by Dolan and Osborn [4] that certain elliptic hypergeometric integrals coincide with superconformal indices of four-dimensional gauge field theories and corresponding identities prove Seiberg dualities (electro-magnetic, strong-weak, or mirror symmetry dualities) in the topological sector. Further detailed investigation of this relationship was performed in many papers among which we mention only a small fraction [5, 6, 7, 8].

The theory of elliptic hypergeometric functions is nowadays a rich mathematical subject with many beautiful new constructions [9]. One of its key ingredients is the elliptic gamma function of order one related to the Barnes multiple gamma function of order three $\Gamma_3(u;\omega_1,\omega_2,\omega_3)$ [10]. The plain and q-hypergeometric functions are directly related to the Barnes gamma functions of order one $\Gamma_1(u;\omega_1)$ (which is proportional to the standard Euler's gamma function $\Gamma(u/\omega_1)$) and order two $\Gamma_2(u;\omega_1,\omega_2)$, respectively. The multiple infinite q-products with several bases naturally emerge in the considerations of superconformal indices for higher dimensional theories [11, 12]. In particular, topological strings partition function [12, 13, 14] and $6d$ superconformal indices [15, 16, 17] are expressed in terms of the elliptic gamma

function of order two (related to $\Gamma_4(u;\omega)$).[1] In [18] the elliptic gamma function of order three (related to $\Gamma_5(u;\omega)$) emerged in the description of partition functions of solvable $2d$ statistical mechanics models identical with superconformal indices of some $4d$ quiver gauge theories.

Shortly after the discovery of elliptic beta integrals the author posed a natural question: is there a higher order generalization of elliptic hypergeometric integrals to Barnes multiple gamma functions $\Gamma_m(u;\omega)$ with $m>3$, obeying exact formulas similar to the ones found in [2] ? Until now this question has not been resolved, but the author is sure it has a positive answer. Perhaps $6d$ superconformal theories provide an appropriate framework for approaching this problem through the systematic investigation of identities for corresponding indices.

In this note we discuss a particular question of the theory of elliptic gamma functions of order two and related $6d$ superconformal indices. We speculate also on the structure of a boundary $6d/4d$ (or $5d/3d$) theory with the exact $W(E_7)$-invariant superconformal index (or partition function).

2. Modified elliptic gamma functions

Let us take four incommensurate quasiperiods $\omega_k\in\mathbb{C}$ (i.e., they are constrained by the condition $\sum_{k=1}^4 n_k\omega_k\neq 0$, $n_k\in\mathbb{Z}$). Using their ratios we form six bases

$$\begin{aligned} &q=e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}},\quad p=e^{2\pi \mathrm{i}\frac{\omega_3}{\omega_2}},\quad r=e^{2\pi \mathrm{i}\frac{\omega_3}{\omega_1}},\\ &s=e^{2\pi \mathrm{i}\frac{\omega_4}{\omega_1}},\quad t=e^{2\pi \mathrm{i}\frac{\omega_4}{\omega_2}},\quad w=e^{2\pi \mathrm{i}\frac{\omega_3}{\omega_4}} \end{aligned} \tag{1}$$

and their particular modular transformations

$$\begin{aligned} &\tilde q=e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}},\quad \tilde p=e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_3}},\quad \tilde r=e^{-2\pi \mathrm{i}\frac{\omega_1}{\omega_3}},\\ &\tilde s=e^{-2\pi \mathrm{i}\frac{\omega_1}{\omega_4}},\quad \tilde t=e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_4}},\quad \tilde w=e^{-2\pi \mathrm{i}\frac{\omega_4}{\omega_3}}. \end{aligned} \tag{2}$$

The bases p,q,r and $\tilde p,\tilde q,\tilde r$ coincide with those used in [2, 9].

In the increasing order of complexity we define the following infinite products all of which are well defined only when bases $q,\ldots,w$ are of modulus less than 1. Denote

$$(z;q_1,\ldots,q_m)=\prod_{k_1,\ldots,k_m=0}^{\infty}(1-zq_1^{k_1}\cdots q_m^{k_m}),\qquad z\in\mathbb{C},$$

the standard infinite q-product and

$$\theta(z;p)=(z;p)(pz^{-1};p)$$

a theta function obeying properties, $\theta(pz;p)=\theta(z^{-1};p)=-z^{-1}\theta(z;p)$. The standard (order one) elliptic gamma function has the form [9]

$$\Gamma(z;p,q)=\prod_{i,j=0}^{\infty}\frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j},\qquad z\in\mathbb{C}^*,$$

and the elliptic gamma function of order two is

$$\Gamma(z;p,q,t)=\prod_{i,j,k=0}^{\infty}(1-z^{-1}p^{i+1}q^{j+1}t^{k+1})(1-zp^iq^jt^k),\qquad z\in\mathbb{C}^*.$$

[1] The author had a joint project with F. Dolan and G. Vartanov on building $6d$ indices where emergence of this function was noticed as well. Unfortunately, it was terminated due to insuperable circumstances.

We use the conventions

$$\Gamma(a,b;\ldots) := \Gamma(a;\ldots)\Gamma(b;\ldots), \quad \Gamma(az^{\pm 1};\ldots) := \Gamma(az;\ldots)\Gamma(az^{-1};\ldots),$$
$$\Gamma(az^{\pm 1}y^{\pm 1};\ldots) := \Gamma(azy;\ldots)\Gamma(az^{-1}y;\ldots)\Gamma(azy^{-1};\ldots)\Gamma(az^{-1}y^{-1};\ldots).$$

Both, $\Gamma(z;p,q)$ and $\Gamma(z;p,q,t)$ are symmetric in their bases. For the order one elliptic gamma function one has the difference equations

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q), \qquad \Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q).$$

For the second order function one has

$$\frac{\Gamma(qz;p,q,t)}{\Gamma(z;p,q,t)} = \Gamma(z;p,t), \quad \frac{\Gamma(pz;p,q,t)}{\Gamma(z;p,q,t)} = \Gamma(z;q,t), \quad \frac{\Gamma(tz;p,q,t)}{\Gamma(z;p,q,t)} = \Gamma(z;p,q).$$

The inversion relations have the form

$$\Gamma(z,pqz^{-1};p,q) = 1, \qquad \Gamma(pqtz;p,q,t) = \Gamma(z^{-1};p,q,t).$$

In [2] the following modified elliptic gamma function of order one was defined

$$G(u;\omega_1,\omega_2,\omega_3) := \Gamma(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};p,q)\Gamma(re^{-2\pi \mathrm{i}\frac{u}{\omega_1}};r,\tilde{q}) = \frac{\Gamma(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};p,q)}{\Gamma(\tilde{q}e^{2\pi \mathrm{i}\frac{u}{\omega_1}};r,\tilde{q})}.$$

It satisfies three linear difference equations of the first order

$$G(u+\omega_1;\omega) = \theta(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};p)G(u;\omega), \tag{3}$$
$$G(u+\omega_2;\omega) = \theta(e^{2\pi \mathrm{i}\frac{u}{\omega_1}};r)G(u;\omega), \tag{4}$$
$$G(u+\omega_3;\omega) = e^{-\pi \mathrm{i} B_{2,2}(u;\omega)}G(u;\omega), \tag{5}$$

where $B_{2,2}(u;\omega)$ is the diagonal Bernoulli polynomial of order two,

$$B_{2,2}(u;\omega) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}.$$

In (5) the exponential coefficient emerged through the following well-known $SL(2,\mathbb{Z})$-modular transformation property of theta functions

$$\frac{\theta\left(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};e^{2\pi \mathrm{i}\frac{\omega_1}{\omega_2}}\right)}{\theta\left(e^{-2\pi \mathrm{i}\frac{u}{\omega_1}};e^{-2\pi \mathrm{i}\frac{\omega_2}{\omega_1}}\right)} = e^{-\pi \mathrm{i} B_{2,2}(u;\omega)}. \tag{6}$$

One has the reflection equation

$$G(a;\omega)G(b;\omega) = 1, \quad a+b = \sum_{k=1}^{3}\omega_k.$$

We shall use below the following shorthand notation

$$G(a\pm b;\omega) := G(a+b,a-b;\omega) := G(a+b;\omega)G(a-b;\omega).$$

In order to prove that (3) is well defined for $|q|=1$ we consider another function

$$\tilde{G}(u;\omega_1,\omega_2,\omega_3) = e^{-\frac{\pi \mathrm{i}}{3}B_{3,3}(u;\omega)}\Gamma(e^{-2\pi \mathrm{i}\frac{u}{\omega_3}};\tilde{r},\tilde{p}), \tag{7}$$

where $B_{3,3}(u;\omega)$ is the diagonal Bernoulli polynomial of order three,

$$B_{3,3}\left(u+\sum_{n=1}^{3}\frac{\omega_n}{2};\omega\right) = \frac{u(u^2-\frac{1}{4}\sum_{k=1}^{3}\omega_k^2)}{\omega_1\omega_2\omega_3}.$$

Obviously, one has the symmetry $\tilde{G}(u;\omega_1,\omega_2,\omega_3) = \tilde{G}(u;\omega_2,\omega_1,\omega_3)$. Using the relation

$$B_{3,3}(u+\omega_3;\omega_1,\omega_2,\omega_3) - B_{3,3}(u;\omega_1,\omega_2,\omega_3) = 3\omega_3 B_{2,2}(u;\omega_1,\omega_2),$$

it is not difficult to check that $\tilde{G}(u;\omega)$ satisfies the same three equations (3)-(5) and the normalization condition

$$\tilde{G}(\frac{1}{2}\sum_{k=1}^{3}\omega_k;\omega_1,\omega_2,\omega_3) = G(\frac{1}{2}\sum_{k=1}^{3}\omega_k;\omega_1,\omega_2,\omega_3) = 1.$$

Therefore by the Jacobi theorem one obtains the equality

$$\tilde{G}(u;\omega_1,\omega_2,\omega_3) = G(u;\omega_1,\omega_2,\omega_3) \tag{8}$$

corresponding to one of the $SL(3,\mathbb{Z})$-modular group transformation laws for the elliptic gamma function [19].

The crucial property of $G(u;\omega)$ is that it remains a well defined meromorphic function of u even for $\omega_1/\omega_2 > 0$ (i.e., for $|q| = 1$ with the conditions $|p|, |r| < 1$ being obligatory), in difference from $\Gamma(z;p,q)$. This fact is evident from the second form of representation of $G(u;\omega)$ (7).

Take the limit $\omega_3 \to \infty$ in such a way that $\mathrm{Im}(\omega_3/\omega_1), \mathrm{Im}(\omega_3/\omega_2) \to +\infty$ (i.e., $p, r \to 0$). Then,

$$\lim_{p,r\to 0} G(u;\omega_1,\omega_2,\omega_3) = \gamma(u;\omega_1,\omega_2) = \frac{(e^{2\pi \mathrm{i} u/\omega_1}\tilde{q};\tilde{q})}{(e^{2\pi \mathrm{i} u/\omega_2};q)}. \tag{9}$$

This is a modified q-gamma function known under many other different names (double sine, hyperbolic gamma function, or quantum dilogarithm, see Appendix A in [18] for interconnections between these functions). For $\mathrm{Re}(\omega_1), \mathrm{Re}(\omega_2) > 0$ and $0 < \mathrm{Re}(u) < \mathrm{Re}(\omega_1+\omega_2)$ it has the integral representation

$$\gamma(u;\omega_1,\omega_2) = \exp\left(-\int_{\mathbb{R}+\mathrm{i}0} \frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})}\frac{dx}{x}\right), \tag{10}$$

which shows that $\gamma(u;\omega_1,\omega_2)$ is meromorphic even for $\omega_1/\omega_2 > 0$, when $|q| = 1$ and the infinite product representation (9) is not applicable.

Euler's gamma function $\Gamma(x)$ can be defined as a special solution of the functional equation $f(x+1) = xf(x)$. q-Gamma functions with $q = e^{2\pi \mathrm{i}\omega_1/\omega_2}$ can be defined as special solutions of the equation $f(x+\omega_1) = (1-e^{2\pi \mathrm{i} u/\omega_2})f(x)$ (in particular the functions (10) and $1/(e^{2\pi \mathrm{i} u/\omega_2};q)$ satisfy this equation). Analogously, the elliptic gamma functions of order one are defined as special solutions of the key equation (3), which does not assume any restriction on the parameter q. Its particular solutions $\Gamma(e^{2\pi \mathrm{i} u/\omega_2};p,q)$ and $1/\Gamma(q^{-1}e^{2\pi \mathrm{i} u/\omega_2};p,q^{-1})$ exist only for $|q| < 1$ or $|q| > 1$, respectively. And $G(u;\omega)$ covers the remaining domain $|q| = 1$.

Define now the modified elliptic gamma function of order two

$$G(u;\omega_1,\ldots,\omega_4) := \frac{\Gamma(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};q,p,t)}{\Gamma(\tilde{q}e^{2\pi \mathrm{i}\frac{u}{\omega_1}};\tilde{q},r,s)}. \tag{11}$$

This is a meromorphic function of $u \in \mathbb{C}$ satisfying the inversion relation

$$G(u+\textstyle\sum_{k=1}^{4}\omega_k;\omega_1,\ldots,\omega_4) = G(-u;\omega_1,\ldots,\omega_4)$$

and four linear difference equations of the first order

$$G(u+\omega_1;\omega)=\Gamma(e^{2\pi\mathrm{i}\frac{u}{\omega_2}};p,t)G(u;\omega), \tag{12}$$

$$G(u+\omega_2;\omega)=\Gamma(e^{2\pi\mathrm{i}\frac{u}{\omega_1}};r,s)G(u;\omega), \tag{13}$$

$$G(u+\omega_3;\omega)=\frac{\Gamma(e^{2\pi\mathrm{i}\frac{u}{\omega_2}};q,t)}{\Gamma(\tilde{q}e^{2\pi\mathrm{i}\frac{u}{\omega_1}};\tilde{q},s)}G(u;\omega), \tag{14}$$

$$G(u+\omega_4;\omega)=\frac{\Gamma(e^{2\pi\mathrm{i}\frac{u}{\omega_2}};p,q)}{\Gamma(\tilde{q}e^{2\pi\mathrm{i}\frac{u}{\omega_1}};\tilde{q},r)}G(u;\omega). \tag{15}$$

Note that the latter equation coefficient is simply $G(u;\omega_1,\omega_2,\omega_3)$. Note also that in the limit $\omega_4\to\infty$ taken in such a way that $s,t\to 0$, we have

$$\lim_{s,t\to 0}G(u;\omega_1,\ldots,\omega_4)=\prod_{j,k=0}^{\infty}\frac{1-e^{2\pi\mathrm{i}\frac{u}{\omega_2}}p^jq^k}{1-e^{2\pi\mathrm{i}\frac{u}{\omega_1}}r^j\tilde{q}^{k+1}},$$

which is only “a half” of $1/G(u;\omega_1,\omega_2,\omega_3)$.

Let us demonstrate that $G(u;\omega_1,\ldots,\omega_4)$ remains a meromorphic function of u for $\omega_1/\omega_2>0$ (or $|q|=1$). First, we find another solution of the above set of equations. Consider the following function

$$\tilde{G}(u;\omega_1,\ldots,\omega_4)=e^{-\frac{\pi\mathrm{i}}{12}B_{4,4}(u;\omega)}\frac{\Gamma(e^{-2\pi\mathrm{i}\frac{u}{\omega_3}};\tilde{p},\tilde{r},\tilde{w})}{\Gamma(we^{-2\pi\mathrm{i}\frac{u}{\omega_4}};\tilde{s},\tilde{t},w)}, \tag{16}$$

where $B_{4,4}(u;\omega)$ is the diagonal multiple Bernoulli polynomial of order four, whose compact form we have found from (37) as

$$\begin{aligned}B_{4,4}(u;\omega_1,\ldots,\omega_4)=\frac{1}{\omega_1\omega_2\omega_3\omega_4}\Big([(u-\frac{1}{2}\sum_{k=1}^{4}\omega_k)^2-\frac{1}{4}\sum_{k=1}^{4}\omega_k^2]^2\\ -\frac{1}{30}\sum_{k=1}^{4}\omega_k^4-\frac{1}{12}\sum_{1\le j<k\le 4}\omega_j^2\omega_k^2\Big).\end{aligned} \tag{17}$$

This function satisfies four linear difference equations of the first order

$$\tilde{G}(u+\omega_1;\omega)=e^{-\frac{\pi\mathrm{i}}{3}B_{3,3}(u;\omega_2,\omega_3,\omega_4)}\frac{\Gamma(e^{-2\pi\mathrm{i}\frac{u}{\omega_3}};\tilde{p},\tilde{w})}{\Gamma(we^{-2\pi\mathrm{i}\frac{u}{\omega_4}};\tilde{t},w)}\tilde{G}(u;\omega), \tag{18}$$

$$\tilde{G}(u+\omega_2;\omega)=e^{-\frac{\pi\mathrm{i}}{3}B_{3,3}(u;\omega_1,\omega_3,\omega_4)}\frac{\Gamma(e^{-2\pi\mathrm{i}\frac{u}{\omega_3}};\tilde{r},\tilde{w})}{\Gamma(we^{-2\pi\mathrm{i}\frac{u}{\omega_4}};\tilde{s},w)}\tilde{G}(u;\omega), \tag{19}$$

$$\tilde{G}(u+\omega_3;\omega)=e^{-\frac{\pi\mathrm{i}}{3}B_{3,3}(u;\omega_1,\omega_2,\omega_4)}\Gamma(e^{-2\pi\mathrm{i}\frac{u}{\omega_4}};\tilde{s},\tilde{t})\tilde{G}(u;\omega), \tag{20}$$

$$\tilde{G}(u+\omega_4;\omega)=e^{-\frac{\pi\mathrm{i}}{3}B_{3,3}(u;\omega_1,\omega_2,\omega_3)}\Gamma(e^{-2\pi\mathrm{i}\frac{u}{\omega_3}};\tilde{p},\tilde{r})\tilde{G}(u;\omega), \tag{21}$$

following from the previously defined formulas and the relation $B_{4,4}(u+\omega_4;\omega)-B_{4,4}(u;\omega)=4\omega_4B_{3,3}(u;\omega)$. But this is precisely the set of equations (12)-(15). Indeed, equality of coefficients in (15) and (21) is nothing else than the relation (8). Equality of coefficients in (12) and (18) or in (14) and (20) follows from (8) after the replacement $\omega_1\to\omega_4$ or $\omega_3\to\omega_4$, respectively. Equality of coefficients in (13) and (19) follows after the replacement in (8) $\omega_1\to\omega_4$ and subsequent substitution $\omega_2\to\omega_1$. Since ω_j's are incommensurate we conclude that the ratio $\tilde{G}(u;\omega)/G(u;\omega)$ is a constant independent on u. However, there is no distinguished

value of u for which the equality of normalizations of G and $\tilde{G}$ becomes obvious. The fact that

$$\tilde{G}(u;\omega_1,\omega_2,\omega_3,\omega_4)=G(u;\omega_1,\omega_2,\omega_3,\omega_4) \tag{22}$$

follows from an $SL(4,\mathbb{Z})$-modular group transformation law for the double elliptic gamma function established as Corollary 9 in [20].[2]

So, in the same way as in the lower order cases, special solutions of the key equation (12) define elliptic gamma functions of order two: the functions $\Gamma(e^{2\pi \mathrm{i}\frac{u}{\omega_2}};p,q,t)$ and $1/\Gamma(q^{-1}e^{2\pi \mathrm{i}\frac{u}{\omega_2}};p,q^{-1},t)$ satisfy it for $|q|<1$ and $|q|>1$, respectively, and $G(u;\omega_1,\ldots,\omega_4)$ covers the domain $|q|=1$. The latter function is defined for $|p|,|r|,|s|,|t|,|w|<1, |q|\leq 1$. Note that for other admissible domains of values of bases it will take a different form.

3. A $6d/4d$ theory with $W(E_7)$-invariant superconformal index

Superconformal indices are defined as [21, 22]

$$I(\underline{y})=\mathrm{Tr}\,[(-1)^F\prod_{k=1}^m y_k^{G_k}e^{-\beta H}],$$

where F is the fermion number, G_k form the maximal Cartan subalgebra preserving a distinguished supersymmetry relations involving one supercharge and its superconformal partner

$$\{Q,S\}=2H, \qquad Q^2=S^2=0, \quad [Q,G_k]=[S,G_k]=0.$$

The trace is effectively taken over the space of BPS states formed by zero modes of the operator H which eliminates dependence on the chemical potential β. Considering supersymmetric theories in curved backgrounds and their indices one comes substantially to the same objects, i.e. one can use them for a description of non-superconformal theories [23]. Computation of such indices via the localization techniques was initiated in [24].

We shall not discuss general structure of these indices in $4d$ field theories since they were described in many previous papers, see, e.g., [6, 7]. Take a particular $4d$ theory in the space-time $S^3\times S^1$ with $SP(2N)$ gauge group and the flavor group $SU(8)\times U(1)$. In addition to the vector superfield in the adjoint representation of $SP(2N)$, take 8 chiral matter fields forming the fundamental representation of $SP(2N)$ with the R-charge $1/2$ and $U(1)$-charge $(1-N)/4$. Take also one antisymmetric $SP(2N)$-tensor field of zero R- and $SU(8)$-charges and unit $U(1)$-charge. For $N=1$, the global group $U(1)$ decouples and the tensor field is absent.

The superconformal index of this theory is described by the following elliptic hypergeometric integral [5]:

$$I(y_1,\ldots,y_8;t;p,q)=\frac{(p;p)^N_\infty(q;q)^N_\infty}{2^N N!}\Gamma(t;p,q)^{N-1}\int_{\mathbb{T}^N}\prod_{1\leq j<k\leq N}\frac{\Gamma(tz_j^{\pm1}z_k^{\pm1};p,q)}{\Gamma(z_j^{\pm1}z_k^{\pm1};p,q)}$$
$$\times\prod_{j=1}^N\frac{\prod_{i=1}^8\Gamma(t^{\frac{1-N}{4}}(pq)^{\frac14}y_iz_j^{\pm1};p,q)}{\Gamma(z_j^{\pm2};p,q)}\frac{dz_j}{2\pi \mathrm{i}z_j}. \tag{23}$$

[2] This transformation was used in [14], though only $SL(3,\mathbb{Z})$-group is explicitly mentioned there. Actually, the $SL(3,\mathbb{Z})$-modular group emerged earlier as a symmetry transformation of $4d$ superconformal indices explaining the 't Hooft anomaly matching conditions [6].

Here y_i are fugacities for $SU(8)$-group satisfying the constraint $\prod_{i=1}^8 y_i = 1$, t is the fugacity for the group $U(1)$, p and q are fugacities for the superconformal group generator combinations $R/2+\overline{J}_3 \pm J_3$, where R is the R-charge. Nontrivial contributions to the index come only from the states with $H = E - 2\overline{J}_3 - 3R/2 = 0$. In terms of the variables $t_i = t^{\frac{1-N}{4}}(pq)^{\frac14} y_i$ we have the balancing condition $t^{2N-2}\prod_{i=1}^8 t_i = (pq)^2$. The constraints $|t|, |t_i| < 1$ are needed for the choice of the integration contours as unit circles with positive orientation $\mathbb{T}$. For $N=1$ the integral $I(y_1,\ldots,y_8;p,q)$ is nothing else than an elliptic analogue of the Euler-Gauss hypergeometric function introduced in [2].

In addition to the obvious S_8-symmetry in variables y_i, function (23) obeys the following hidden symmetry transformation extending S_8-group to $W(E_7)$ – the Weyl group of the exceptional root system E_7:

$$
\begin{aligned}
I(y_1,\ldots,y_8;t;p,q) = \prod_{k=0}^{N-1}\Big(&\prod_{1\le i<j\le 4}\Gamma(t^{k+\frac{1-N}{2}}(pq)^{\frac12}y_iy_j;p,q)\\
&\times\prod_{5\le i<j\le 8}\Gamma(t^{k+\frac{1-N}{2}}(pq)^{\frac12}y_iy_j;p,q)\Big)I(\hat{y}_1,\ldots,\hat{y}_8;t;p,q),
\end{aligned}
$$

where

$$
\hat{y}_k = \frac{y_k}{\sqrt{Y}}, \qquad \hat{y}_{k+4} = \sqrt{Y}y_{k+4}, \quad k=1,\ldots,4, \qquad Y = y_1y_2y_3y_4.
$$

Equivalently one can write $Y^{-1} = y_5y_6y_7y_8$. For $N=1$ this relation was established by the author [2] and it was extended to arbitrary N by Rains [25].

Define now the following expression containing the double elliptic gamma function

$$
I_{6d/4d}(y_1,\ldots,y_8;t;p,q) := \frac{I(y_1,\ldots,y_8;t;p,q)}{\prod_{1\le j<k\le 8}\Gamma(t^{\frac{N+1}{2}}(pq)^{\frac12}y_jy_k;p,q,t)}. \tag{24}
$$

First, we show that this function is $W(E_7)$-group invariant. Indeed, explicit substitution yields

$$
I_{6d/4d}(\hat{y}_1,\ldots,\hat{y}_8;t;p,q) = I_{6d/4d}(y_1,\ldots,y_8;t;p,q), \tag{25}
$$

which follows from the relation

$$
\begin{aligned}
\prod_{1\le j<k\le 8}\frac{\Gamma(t^{\frac{N+1}{2}}(pq)^{\frac12}y_jy_k;p,q,t)}{\Gamma(t^{\frac{N+1}{2}}(pq)^{\frac12}\hat{y}_j\hat{y}_k;p,q,t)} &= \prod_{\substack{1\le j<k\le 4\\ 5\le j<k\le 8}}\frac{\Gamma(t^{\frac{N+1}{2}}(pq)^{\frac12}y_jy_k;p,q,t)}{\Gamma(t^{\frac{-N+1}{2}}(pq)^{\frac12}y_jy_k;p,q,t)}\\
&= \prod_{k=0}^{N-1}\prod_{\substack{1\le j<k\le 4\\ 5\le j<k\le 8}}\Gamma(t^{k+\frac{1-N}{2}}(pq)^{\frac12}y_jy_k;p,q).
\end{aligned}
$$

Now we want to interpret equality (25) as a symmetry relation for the superconformal index of a $6d$ field theory similar to [8], where a beautiful $5d$ field theory interpretation was given to the $W(E_7)$-symmetry of the elliptic analogue of Euler-Gauss hypergeometric function.

The $6d$-index is defined as

$$
I(\underline{y};p,q,t) = \mathrm{Tr}\,[(-1)^F p^{C_1}q^{C_2}t^{C_3}\prod_k y_k^{G_k}],
$$

where G_k are the flavor group maximal torus generators and $C_{1,2,3}$ are Cartans for the space-time symmetry group. In the notations of Imamura [16]

$$p^{C_1}q^{C_2}t^{C_3}=x^{j_1+3R/2}y_3^{j_2}y_8^{j_3},$$

where R is the Cartan of $6d$ (1,0)-supersymmetry $SU(2)_R$-subalgebra, j_1 is the generator of $U(1)_V$ and j_2, j_3 are Cartans of $SU(3)_V$ with $U(1)_V\times SU(3)_V$ being the isometry subgroup of Lorentz rotations in the $S^5\times S^1$ space. Perturbative contributions to the index are described by the double elliptic gamma function [16, 17]. E.g., for a $U(1)$-flavor group hypermultiplet one has

$$\begin{aligned} I_{hyp}(y;p,q,t) &= \frac{1}{\Gamma(\sqrt{pqt}y;p,q,t)} = \exp\Big(\sum_{n=1}^\infty \frac{i_{hyp}(y^n;p^n,q^n,t^n)}{n}\Big), \\ i_{hyp}(y;p,q,t) &= \frac{\sqrt{pqt}(y+y^{-1})}{(1-p)(1-q)(1-t)}. \end{aligned} \tag{26}$$

For $SU(2)$ gauge group vector superfield one obtains[3]

$$\begin{aligned} I_{vec}(z;p,q,t) &= \kappa\frac{\Gamma(z^{\pm2};p,q,t)}{(1-z^2)(1-z^{-2})} = \exp\Big(\sum_{n=1}^\infty \frac{i_{vec}(z^n;p^n,q^n,t^n)}{n}\Big), \\ i_{vec}(z;p,q,t) &= \Big(1-\frac{1+pqt}{(1-p)(1-q)(1-t)}\Big)\chi_{adj,\,SU(2)}(z), \\ \kappa &= \lim_{x\to1}\frac{\Gamma(x;p,q,t)}{1-x} = (p;p)(q;q)(t;t)(pq;p,q)(pt;p,t)(qt;q,t)(pqt;p,q,t)^2. \end{aligned} \tag{27}$$

Consider the following six-dimensional space $S^3\times S^1\times S^2$. Associate fugacities p and q with the isometries of $S^3\times S^1$, the space where the above $4d$ gauge theory lives, and the fugacity t with a $U(1)$-group which has the meaning of an isometry of S^2. We stress that in this way the group $U(1)$, which was playing from $4d$ point of view a role of the flavor symmetry subgroup, becomes the space-time symmetry subgroup. Suppose that this manifold reduces after the dimensional reduction associated with the limit $t\to0$ to $5d$ space-time considered in [8] (formally it corresponds to shrinking $S^2\to S^1$). Let us take a gauge invariant $6d$ hypermultiplet forming the totally antisymmetric tensor representation of the mentioned $SU(8)$ flavor group and having the $U(1)$-group charge $N/2$. Its index is

$$I_{T_A}(\underline{y};p,q,t)=\prod_{1\le j<k\le8}\frac{1}{\Gamma(t^{N/2}\sqrt{pqt}y_jy_k;p,q,t)},\quad \prod_{k=1}^8 y_k=1,$$

which evidently coincides with the multiplier in (24). This $6d$ hypermultiplet couples to the $4d$ theory located in a codimension 2 space which gives its own contribution to the superconformal index described by the integral $I(y_1,\ldots,y_8;t;p,q)$. The combined index has $W(E_7)$-symmetry indicating that this theory may have the enhanced E_7-flavor group, provided there exists an appropriate point in the moduli space.

[3]The multiplier κ was skipped in the considerations of [16], but it appears naturally from the adjoint representation character $\chi_{adj,\,SU(2)}(z)=z^2+z^{-2}+1$. One can incorporate into I_{vec} a piece of the Haar measure for $SU(2)$ and cancel thus the terms $(1-z^2)(1-z^{-2})$.

For $N=1$ a substantial simplification of the situation takes place. The index reduces to

$$I_{6d/4d}(y_1,\ldots,y_8;t;p,q)=\frac{(p;p)(q;q)}{2\prod_{1\le j<k\le 8}\Gamma((pq)^{\frac12}y_j^{-1}y_k^{-1};p,q,t)}$$
$$\times\int_{\mathbb{T}}\frac{\prod_{k=1}^8\Gamma((pq)^{\frac14}y_kz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{2\pi\mathrm{i}z}, \tag{28}$$

where we have used the inversion relation for the double elliptic gamma function. Since ω_4^{-1} can be considered as a squashing parameter for deforming S^2, the limit $\omega_4\to\infty$ shrinks our $6d$ space-time to the $5d$ space-time considered in [8], namely, $D^4\times S^1$, where $D=S^3\times S^1$ with S^3 being a boundary of a half S^4-sphere cut along its equator. Note that although the $U(1)$-group is completely decoupled from the $4d$ theory, the index $I_{6d/4d}$ still keeps track of this background symmetry.

This interpretation gets a confirmation from the $6d/4d$-index which reduces for $t\to 0$ precisely to the corresponding $5d/4d$ half-index of [8]:

$$I_{6d/4d}(y_1,\ldots,y_8;0;p,q)=I_{5d/4d}(y_1,\ldots,y_8;p,q) \tag{29}$$
$$=\frac{(p;p)(q;q)}{\prod_{1\le j<k\le 8}\prod_{n,m=0}^\infty(1-y_j^{-1}y_k^{-1}p^{n+\frac12}q^{m+\frac12})}\int_{\mathbb{T}}\frac{\prod_{k=1}^8\Gamma((pq)^{\frac14}y_kz^{\pm1};p,q)}{\Gamma(z^{\pm2};p,q)}\frac{dz}{4\pi\mathrm{i}z},$$

As proposed in [2], one can build elliptic hypergeometric integrals using the modified elliptic gamma function. This is achieved by mere replacement of $\Gamma(e^{2\pi\mathrm{i}u/\omega_2};p,q)$-functions by $G(u;\omega_1,\omega_2,\omega_3)$ and appropriate change of the integration contour. Taking the limit $\omega_3\to\infty$ such that $p,r\to 0$ one obtains hyperbolic hypergeometric integrals expressed in terms of the hyperbolic gamma function $\gamma^{(2)}(u;\omega_1,\omega_2)$ (see the Appendix). Using this procedure the integral (23) was reduced in [26, 27] to the following expression:

$$I_h(x_1,\ldots,x_8;\lambda;\omega_1,\omega_2)=\frac{1}{2^NN!}\gamma^{(2)}(\lambda;\omega_1,\omega_2)^{N-1}$$
$$\times\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\prod_{1\le j<k\le N}\frac{\gamma^{(2)}(\lambda\pm u_j\pm u_k;\omega_1,\omega_2)}{\gamma^{(2)}(\pm u_j\pm u_k;\omega_1,\omega_2)}\prod_{j=1}^N\frac{\prod_{k=1}^8\gamma^{(2)}(\mu_k\pm u_j;\omega_1,\omega_2)}{\gamma^{(2)}(\pm 2u_j;\omega_1,\omega_2)}\prod_{j=1}^N\frac{du_j}{\mathrm{i}\sqrt{\omega_1\omega_2}},$$

where chemical potentials are related to flavor fugacities as $t=e^{2\pi\mathrm{i}\lambda/\omega_2}$ and $y_k=e^{2\pi\mathrm{i}x_k/\omega_2}$ with

$$\mu_k=x_k+\frac{\omega_1+\omega_2}{4}-(N-1)\frac{\lambda}{4},\qquad \sum_{k=1}^8x_k=0.$$

In terms of μ_k the balancing condition reads

$$2(N-1)\lambda+\sum_{k=1}^8\mu_k=2(\omega_1+\omega_2).$$

Define now a new function

$$I_{5d/3d}(x_1,\ldots,x_8;\lambda;\omega_1,\omega_2)=\frac{I_h(x_1,\ldots,x_8;\lambda;\omega_1,\omega_2)}{\prod_{1\le j<k\le 8}\gamma^{(3)}(\frac{N+1}{2}\lambda+\mu_j+\mu_k;\omega_1,\omega_2,\lambda)}, \tag{30}$$

where $\gamma^{(3)}(u;\omega_1,\omega_2,\lambda)$ is the hyperbolic gamma function of third order (see the Appendix). Again, it is not difficult to see that this function is $W(E_7)$-invariant as

a consequence of known relations for I_h-integral:

$$I_{5d/3d}(\hat{x}_1,\ldots,\hat{x}_8;\lambda;\omega_1,\omega_2)=I_{5d/3d}(x_1,\ldots,x_8;\lambda;\omega_1,\omega_2), \tag{31}$$

where

$$\hat{x}_k=x_k-\frac{1}{2}\sum_{l=1}^4 x_l,\qquad \hat{x}_{k+4}=x_{k+4}+\frac{1}{2}\sum_{l=1}^4 x_l,\quad k=1,\ldots,4.$$

Integral (30) can be interpreted as the partition function of a particular boundary $5d/3d$-theory. Indeed, contribution of $5d$ hypermultiplets to the partition function is determined by the $1/\gamma^{(3)}$-function [14, 16]. This fact indicates that our $5d$ theory has the same boundary geometry as the one described earlier with the boundary $S^3\times S^1$ replaced by the squashed three-sphere S_b^3 with $b^2=\omega_1/\omega_2$ and preserved initial S^2.

The symmetry transformation (31) for partition functions of particular $5d$ theories with 8 hypermultiplets and specific $3d$ boundary conditions is related to the theories with enhanced E_n-global symmetries discussed in [28]. Definitely, a more detailed physical investigation of discussed $6d$ and $5d$ models is required, which is a matter of a separate consideration. Namely, it is necessary to show that the moduli spaces of these theories contain points with the exact E_7-flavor groups and make the connection with the models of [28] more explicit.

4. Relevance of modular group transformations

Let us discuss physical relevance of modular groups acting on generalized gamma functions. Quasiperiods ω_k are usually interpreted as squashing parameters and coupling constants. The generalized gamma functions are defined differently for different domains of these parameters related to each other by modular transformations usually playing the role of S-dualities.

The simplest example of the relevance of $SL(2,\mathbb{Z})$-modular group is given by the q-gamma function. It can be defined as a solution of the equation $f(u+\omega_1)=(1-e^{2\pi\mathrm{i}u/\omega_2})f(u)$. For $|q|<1$ its solution $1/(e^{2\pi\mathrm{i}u/\omega_2};q)$ defines the standard q-gamma function and it serves as a building block of various partition functions. However, to cover the region $|q|=1$, one needs $SL(2,\mathbb{Z})$-modular transformation and define the modified q-gamma function

$$\gamma(u;\omega_1,\omega_2)=\frac{(\tilde{q}e^{2\pi\mathrm{i}u/\omega_1};\tilde{q})}{(e^{2\pi\mathrm{i}u/\omega_2};q)},$$

i.e. to use the ratio of modular transformed elementary partition functions. A list of references to works dedicated to this function in its various instances can be found in Appendix A of [18].

Consider now the elliptic gamma function of order one $\Gamma(z;p,q)$ describing the superconformal index for a $4d$ chiral superfield. In order to define an analogue of this function for the region $|q|=1$ in [2] the modified elliptic gamma function was proposed as the ratio of this index with a $U(1)$-group fugacity parametrization $z=e^{2\pi\mathrm{i}u/\omega_2}$ and superconformal group generator fugacities $q=e^{2\pi\mathrm{i}\omega_1/\omega_2}$ and $p=e^{2\pi\mathrm{i}\omega_3/\omega_2}$ and the index with a different choice of squashing parameters $\Gamma(\tilde{q}e^{2\pi\mathrm{i}\frac{u}{\omega_1}};r,\tilde{q})$. Surprisingly, this ratio yields again the chiral field index with yet another parametrization of fugacities $e^{-\frac{\pi\mathrm{i}}{3}B_{3,3}(u;\omega)}\Gamma(e^{-2\pi\mathrm{i}\frac{u}{\omega_3}};\tilde{r},\tilde{p})$. The exponential cocycle factor spoils this interpretation and requires a physical interpretation. As

shown in [6] this $SL(3,\mathbb{Z})$-group action on $4d$ superconformal indices describes the 't Hooft anomaly matching conditions as the conditions of cancellation of this cocycle contributions described by a curious set of Diophantine equations. Therefore this modular group plays quite important role in the formalism.

A similar picture at the level of free $6d$ hypermultiplet index was described recently in [14] in relation to the topological strings partition function. Namely, $I_{hyp}(y;p,q,t)$ is proportional to the latter function and, as argued in [14], a particular combination of three $SL(4,\mathbb{Z})$-transformed versions of it should yield yet another similar partition function. And this expectation is confirmed with the help of an $SL(4,\mathbb{Z})$-modular group transformation for the double elliptic gamma function which is written in our case as equality (22).

However, in difference from the $G(u;\omega_1,\omega_2,\omega_3)$-function case, the elliptic hypergeometric integrals formed from $G(u;\omega_1,\ldots,\omega_4)$ do not reduce to the integrals composed from $\Gamma(z;p,q,t)$. Now the modular group simply maps them into similar integrals up to the cocycle exponential $\propto e^{-\frac{\pi \mathrm{i}}{12}B_{4,4}}$ multiplying the integral kernels. Therefore one should not expect cancellation of these factors from the integrals. And indeed, the cancellation of even the gauge group chemical potentials is possible only under very strong restrictions. E.g. for $SU(2)$ gauge group it is possible only for $N_f = 16$ at the expense of an unusual quadratic restriction on chemical potentials. Such exponentials have the forms resembling the Casimir energy contributions to the indices [17]. Therefore it is necessary to better understand the general structure of full $6d$ superconformal indices before connecting $SL(4,\mathbb{Z})$-modular group transformations to higher dimensional anomalies. Still, we can see an involvement of the $B_{4,4}$-polynomial in the $4d$ anomaly matching conditions.

Define a modified elliptic hypergeometric integral:

$$\begin{aligned} I^{mod}(x_1,\ldots,x_8;\omega_1,\ldots,\omega_4) &= \frac{(\tilde{p};\tilde{p})^N(\tilde{r};\tilde{r})^N}{2^N N!} G(\omega_4;\omega_1,\omega_2,\omega_3)^{N-1} \\ &\times \int_{[-\frac{\omega_3}{2},\frac{\omega_3}{2}]^N} \prod_{1\leq j<k\leq N} \frac{G(\omega_4 \pm u_j \pm u_k;\omega_1,\omega_2,\omega_3)}{G(\pm u_j \pm u_k;\omega_1,\omega_2,\omega_3)} \\ &\times \prod_{j=1}^N \frac{\prod_{i=1}^8 G(x_i - \frac{N}{4}\omega_4 + \frac14 \sum_{k=1}^4 \omega_k \pm u_j;\omega_1,\omega_2,\omega_3)}{G(\pm 2u_j;\omega_1,\omega_2,\omega_3)} \frac{du_j}{\omega_3}, \end{aligned} \tag{32}$$

which is obtained from (23) simply by the replacement of Γ-functions by G-functions using exponential representation for fugacities in terms of chemical potentials and passing to the integration over a cube. Note that this integral is well-defined for $|q| = 1$. Introduce now the modified index

$$\begin{aligned} &I^{mod}_{6d/4d}(x_1,\ldots,x_8;\omega_1,\ldots,\omega_4) \\ &\qquad = \frac{I^{mod}(x_1,\ldots,x_8;\omega_1,\ldots,\omega_4)}{\prod_{1\leq j<k\leq 8} G(\frac{N}{2}\omega_4 + \frac12 \sum_{k=1}^4 \omega_k + x_j + x_k;\omega_1,\ldots,\omega_4)}, \end{aligned} \tag{33}$$

containing the modified double elliptic gamma function. It is not difficult to check that this index is also $W(E_7)$-invariant

$$I^{mod}_{6d/4d}(\hat{x}_1,\ldots,\hat{x}_8;\omega_1,\ldots,\omega_4) = I^{mod}_{6d/4d}(x_1,\ldots,x_8;\omega_1,\ldots,\omega_4). \tag{34}$$

Now one can replace G-functions by their modular transformed expressions $\tilde{G}$ containing exponentials of Bernoulli polynomials and check that relation (34) boils down to an $SL(3,\mathbb{Z})$-modular transformation of the previous relation (25). At the

level of integral (32) with the constraint $2(x_7+x_8)=\sum_{k=1}^3\omega_k+(N-1)\omega_4$ this was done already in [26]. Importantly, as shown in [6], the condition of cancellation of Bernoulli polynomial coefficients in the integration variables and the external parameters describes the 't Hooft anomaly matching conditions. Therefore the fourth order polynomial $B_{4,4}(u;\omega)$ is effectively involved into these anomaly matchings as well.

The residue calculus for elliptic hypergeometric integrals/$4d$ indices was developed long ago, see [2, 9] and references therein. It has shown that by shrinking the integration contour to zero one can formally represent integrals as sums of bilinear combinations of elliptic hypergeometric series with permuted base variables which describes the factorization of superconformal indices into some more elementary building blocks which in general are not defined in the limit $p\to 0$ or $q\to 0$. This analysis has lead to the discovery of the notion of two-index biorthogonality and the elliptic modular doubling principle [2, 9]. In [7] this residue calculus applied to $4d$ $\mathcal{N}=2$ superconformal indices was physically interpreted as a result of insertions of surface defects into the bulk theory.

One can investigate the structure of residues for the modified elliptic hypergeometric integrals/indices and come to similar factorization in terms of different elliptic hypergeometric series. The latter series are related by an $SL(3,\mathbb{Z})$-transformation and remain well defined in the limit $p\to 0$, which leads to hyperbolic integrals. As a result hyperbolic integrals are represented as combinations of products of two q-hypergeometric series related by an $SL(2,\mathbb{Z})$-modular transformation (their bases are q and $\tilde{q}$) [2, 9]. This factorization was used in [29] for computing partition functions in some $3d$ $\mathcal{N}=2$ theories appearing from the reduction of $4d$ $\mathcal{N}=4$ SYM theories. The principle difference between $4d$ (elliptic) and $3d$ (hyperbolic) cases consists in the fact that in $3d$ this factorization of sums of residues into holomorphic blocks has rigorous meaning because of the convergence of corresponding infinite series for $|q|<1$, whereas in $4d$ such series do not converge for generic values of p and q bases and the factorization of indices has in general a formal meaning. It is not difficult to develop the residue calculus for $6d$ indices and find triple sums of residues. However, corresponding sums cannot factorize because there are no triply periodic functions. This makes the $4d$/elliptic case rather unique and raises the interest to $6d$ indices as more complicated objects.

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Appendix A. Barnes multiple gamma function

Barnes multiple zeta function $\zeta_m(s,u;\omega)$ [10] is originally defined by an m-fold series

$$\zeta_m(s,u;\omega)=\sum_{n_1,\ldots,n_m=0}^{\infty}\frac{1}{(u+\Omega)^s},\qquad \Omega=n_1\omega_1+\ldots+n_m\omega_m,$$

where $s,u\in\mathbb{C}$. It converges for $\mathrm{Re}(s)>m$, provided all ω_j lie in one half-plane formed by a line passing through zero (then there are no accumulation points of the Ω-lattice in compact domains).

This zeta function satisfies equations

$$\zeta_m(s,u+\omega_j;\omega)-\zeta_m(s,u;\omega)=-\zeta_{m-1}(s,u;\omega(j)),\quad j=1,\dots,m, \tag{35}$$

where $\omega(j)=(\omega_1,\dots,\omega_{j-1},\omega_{j+1},\dots,\omega_m)$ and $\zeta_0(s,u;\omega)=u^{-s}$. The multiple gamma function is defined by Barnes as

$$\Gamma_m(u;\omega)=\exp(\partial\zeta_m(s,u;\omega)/\partial s)\big|_{s=0}.$$

As a consequence of (35) it satisfies m finite difference equations

$$\Gamma_m(u+\omega_j;\omega)=\frac{1}{\Gamma_{m-1}(u;\omega(j))}\,\Gamma_m(u;\omega),\qquad j=1,\dots,m, \tag{36}$$

where $\Gamma_0(u;\omega):=u^{-1}$.

The multiple sine-function is defined as

$$S_m(u;\omega)=\frac{\Gamma_m(\sum_{k=1}^m\omega_k-u;\omega)^{(-1)^m}}{\Gamma_m(u;\omega)}$$

and the hyperbolic gamma function is

$$\gamma^{(m)}(u;\omega)=S_m(u;\omega)^{(-1)^{m-1}}.$$

One has equations

$$\gamma^{(m)}(u+\omega_j;\omega)=\gamma^{(m-1)}(u;\omega(j))\,\gamma^{(m)}(u;\omega),\qquad j=1,\dots,m.$$

The elliptic gamma function of order one can be written as a special ratio of four $\Gamma_3(u;\omega)$-functions, and the elliptic gamma function of order two is given by a product of four $\Gamma_4(u;\omega)$-functions [9].

One can derive the integral representation [20]

$$\begin{aligned}\gamma^{(m)}(u;\omega)&=\exp\left(-\mathrm{PV}\int_{\mathbb{R}}\frac{e^{ux}}{\prod_{k=1}^m(e^{\omega_k x}-1)}\frac{dx}{x}\right)\\&=\exp\left(-\frac{\pi\mathrm{i}}{m!}B_{m,m}(u;\omega)-\int_{\mathbb{R}+\mathrm{i}0}\frac{e^{ux}}{\prod_{k=1}^m(e^{\omega_k x}-1)}\frac{dx}{x}\right)\\&=\exp\left(\frac{\pi\mathrm{i}}{m!}B_{m,m}(u;\omega)-\int_{\mathbb{R}-\mathrm{i}0}\frac{e^{ux}}{\prod_{k=1}^m(e^{\omega_k x}-1)}\frac{dx}{x}\right),\end{aligned}$$

where $\mathrm{Re}(\omega_k)>0$ and $0<\mathrm{Re}(u)<\mathrm{Re}(\sum_{k=1}^m\omega_k)$ and $B_{m,m}$ are multiple Bernoulli polynomials defined by the generating function

$$\frac{x^m e^{xu}}{\prod_{k=1}^m(e^{\omega_k x}-1)}=\sum_{n=0}^\infty B_{m,n}(u;\omega_1,\dots,\omega_m)\frac{x^n}{n!}. \tag{37}$$

In particular, one has the following relation with the modified q-gamma function $\gamma(u;\omega_1,\omega_2)$:

$$\gamma^{(2)}(u;\omega_1,\omega_2)=e^{-\frac{\pi\mathrm{i}}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma(u;\omega_1,\omega_2).$$

Collapsing integrals to sums of residues one can derive infinite product representations for $\gamma^{(m)}(u;\omega)$ [20]. Particular inversion relations have the form

$$\begin{aligned}&\gamma^{(2)}(\textstyle\sum_{k=1}^2\omega_k+u;\omega_1,\omega_2)\gamma^{(2)}(-u;\omega_1,\omega_2)=1,\\&\gamma^{(3)}(\textstyle\sum_{k=1}^3\omega_k+u;\omega_1,\omega_2,\omega_3)=\gamma^{(3)}(-u;\omega_1,\omega_2,\omega_3).\end{aligned}$$

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